DATA SCIENCE

Two-Stage Stochastic Programming: Formulation of Markovitz's Model

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New risk measures

- Until now, we have considered risk measures that penalize the deviation (semideviation) from the mean
- In real life portfolio managers are interested to the losses
- "What is the risk that I lose money? How much can I lose? What is the chance that I lose even more?"
- These are typical questions that almost every investor who has invested or is considering investing in a risky asset asks at some point in time
- These considerations are led to the definition of new measures to control the risk of losses

The Value at Risk

- The Value at Risk (VaR) is a measure of the potential risk that a portfolio of financial assets can suffer.
- In particular, the VaR measures the maximum potential loss that a risky asset or portfolio may incur over a defined period of time with a given confidence level.
- If the VaR on an asset is $ 100 at one-week, with a 95% confidence level, there is a only a 5% chance that the value of the asset will drop more than $ 100 over any given week.
- In mathematical terms, indicating with $\widetilde{L}$ the random loss, and by α a given confidence level, the VaR is defined as:

$$P(\widetilde{L} \leq VaR) = \alpha$$

How the Value at Risk can be computed ?

- The VaR is the α- quantile of the distribution function of the random loss.
- For the moment, we shall assume to consider a predefined portfolio and we shall compute the loss as deviation of the portfolio value from a given target K

$$\widetilde{L} = \max(0, K - \widetilde{W})$$

  where $\widetilde{W}$ represents the random wealth
- If we consider continuous distributions, the VaR can be computed in some special cases – for example, for normal distributions –
- Typically, discrete random variables are considered eventually as approximation of the continuous ones
- In this case, the VaR can be easily determined

Computation in the case of discrete distributions?

- We assume to consider S scenarios of prices and to have a predefined portfolio
- Under each scenario s, we compute the wealth W^s and the loss

$$L^s = max(0, K - W^s)$$

  where K represents a given target
- We determine the distribution function of the losses
- The VaR is the α- quantile of the distribution, that is the value of the losses such that

$$F(VaR) = \alpha$$

The Value at Risk: example

Assuming equiprobable scenarios, compute the VaR at 80%

Scenarios	Portfolio A
1	562,29
2	264,89
3	328,37
4	731,47
5	765,95
6	803,76
7	369,36
8	402,79
9	432,44
10	459,93
11	486,17
12	511,71
13	536,98
14	587,93
15	614,18
16	641,32
17	669,66
18	699,56
19	846,00
20	894,25
21	1252,53
22	1552,69
23	951,11
24	1021,11
25	1113,53

The Value at Risk: determine the density function

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1	264,89
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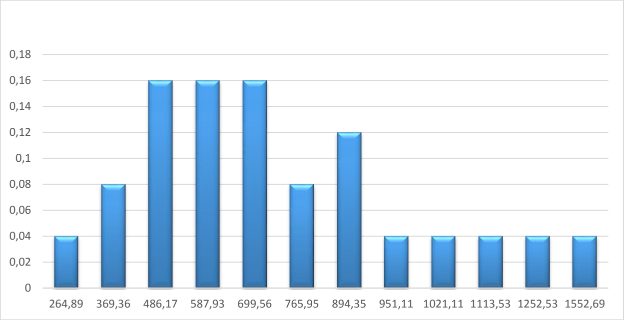
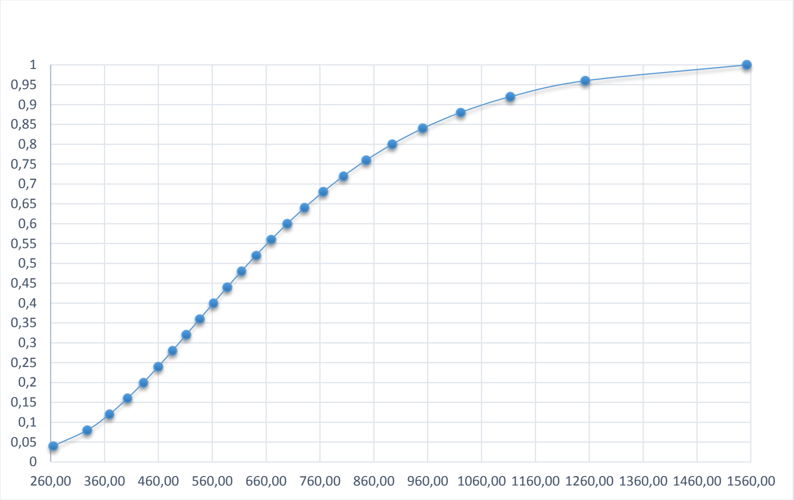
The Value at Risk: determine the density function



The Value at Risk: determine the distribution function



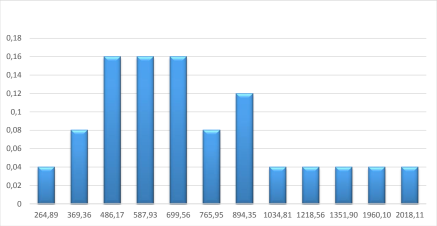
The Value at Risk: remarks

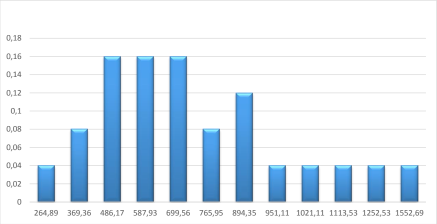
- The VaR was introduced by the investments bank J.P. Morgan in the early 1990s
- It is used often by the banks to capture the potential loss in value of their traded portfolios from adverse market movements over a specified period; this can then be compared to their available capital and cash reserves to ensure that the losses can be covered without putting the firms at risk.
- Since its introduction, the VaR quickly became a widely used measure of risk, because:
  - It summarizes in a single number an important aspect of risk
  - It is easy to understand and is expressed in the best of possible units of measures: LOST MONEY
  - It asks the simple question: How bad can things get?
  - It provides a loss and a probability of occurrence

The Value at Risk: another example

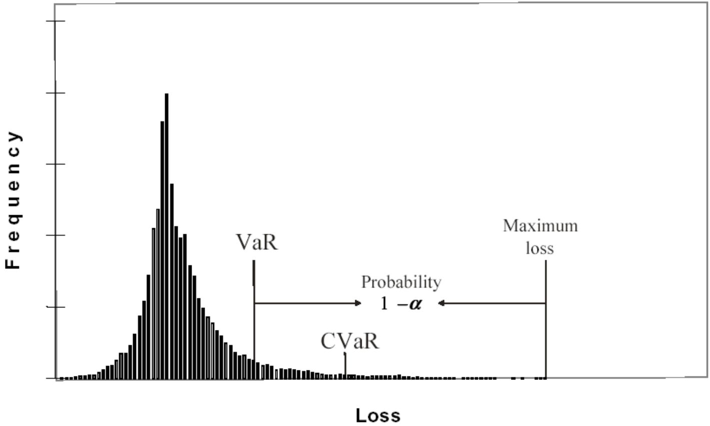
Scenarios	Portfolio A	Portfolio B
1	264,89	264,89
2	328,37	328,37
3	369,36	369,36
4	402,79	402,79
5	432,44	432,44
6	459,93	459,93
7	486,17	486,17
8	511,71	511,71
9	536,98	536,98
10	562,29	562,29
11	587,93	587,93
12	614,18	614,18
13	641,32	641,32
14	669,66	669,66
15	699,56	699,56
16	731,47	731,47
17	765,95	765,95
18	803,76	803,76
19	846,00	846,00
20	894,25	894,25
21	951,11	1034,81
22	1021,11	1218,56
23	1113,53	1351,90
24	1252,53	1960,10
25	1552,69	2018,11

The Value at Risk: compare the VaR of Portfolio A and B





The Value at Risk



The Value at Risk: drawbacks

- The VaR does not care about the losses exceeding the VaR
- Even though the probability of occurrence is limited, the amount of such losses can be very high
- For this reason a new risk measure, named CVaR, has been proposed
- It is defined as the expected value of the losses exceeding the VaR

$$CVaR = E[\tilde{L}|\tilde{L} \geq VaR]$$

Condition Value at Risk: calculation

- As before, we assume that the random prices are discretely distributed
- For each scenario we determine W^s and L^s
- We compute the VaR
- We determine the deviation from the VaR

$$y^s = \max(0, L^s - VaR)$$

- We compute the CVaR as

$$CVaR = VaR + \frac{1}{(1-\alpha)} \sum_{s=1}^{S} p_s y^s$$

- Remark: pay attention to the normalization

Condition Value at Risk: example

- Compute the CVaR of the portfolio A and B
- Even though the two portfolio have the same VaR equal to $894,25$, the CVaR are different

$$CVaR(A) = 1178,194$$

$$CVaR(B) = 1516,696$$

Coherent Risk Measures

- Remark: A risk measure should attribute greater values to more risk investments
- In 1999 Artzner et. al. published a paper where they defined the properties that a good risk measure (termed **coherent**) should have
- They proved that the *standard deviation is not a coherent measure*

Axioms of Coherent Risk Measures

Let us denote by X and Y two random variables representative of the loss of two assets (portfolio)
A risk measure $\rho(x)$ is said to be coherent if it satisfies the following axioms

1. **Subadditivity**

$$\rho(X+Y) \leq \rho(X) + \rho(Y)$$

   The subadditivity axiom captures the meaning of diversification. The risk measure for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged.

2. **Positive homogeneity**

$$\rho(\lambda X) = \lambda \rho(X)$$

   If we change the size of the investment by a factor λ, then the risk measure should be multiplied byλ.

Axioms of Coherent Risk Measures

3 **Translation Invariance**

$$\rho(X + a) = \rho(x) + a$$

If we increase a loss of given amount, its risk measure should go up by a.

4 **Monotonicity** If $X \leq Y$ for every state of the world then

$$\rho(X) \leq \rho(Y)$$

If a portfolio has lower losses than another portfolio for every state of the world, its risk measure should be lower.

Example

We prove that the standard deviation is not a coherent risk measure.
Let us consider the losses associated with two assets X and Y.
Assuming that the scenarios are those reported below, verify if the monotonicity axiom is satisfied.

X	P(X=x)	Y	P(Y=y)
1	0,95	2	0,95
2	0,04	2	0,04
2	0,01	2	0,01

Example

We prove that the standard deviation is not a coherent risk measure.
Let us consider the losses associated with two assets X and Y. Assuming that the scenarios are those reported below, verify if the monotonicity axiom is satisfied.

X	P(X=x)	Y	P(Y=y)
1	0,95	2	0,95
2	0,04	2	0,04
2	0,01	2	0,01

$$E[X] = 1,05 \qquad E[Y] = 2$$

$$\sigma_X^2 = 0,04 \qquad \sigma_Y^2 = 0$$

$$\sigma_X = 0,21 \qquad \sigma_Y = 0$$

$$X \leq Y \quad but \qquad \rho(X) \geq \rho(Y)$$

Exercise

We consider the losses associated with two assets and we assume to know the mass distribution function. Compute the VaR at 0,95 % and prove that it is not a coherent risk measure, since it does not satisfy the subadditivity axiom.

X	P(X=x)
0	0,93
1	0,02
2	0,05

Y	P(Y=y)
0	0,94
0,5	0,01
2,5	0,05

Exercise

Determine the joint probability mass function assuming that X and Y are independent

$$p(x, y) = P(X = x \text{ and } Y = y) = P(X = x) * P(Y = y)$$

X/Y	0	0,5	2,5
0	0,8742	0,0093	0,0465
1	0,0188	0,0002	0,001
2	0,047	0,0005	0,0025

Exercise

Determine the joint distribution function $F(z) = P(X + Y \leq z)$

X+Y	0	0,5	2,5
0	0	0,5	2,5
1	1	1,5	3,5
2	2	2,5	4,5

X+Y	F(X+Y)
0	0,874
0,5	0,884
1	0,902
1,5	0,903
2	0,950
2,5	0,997
3,5	0,998
4,5	1

But

$$\rho(X + Y) = 2 > \rho(X) + \rho(Y) = 1 + 0,5$$

Exercise

Determine the CVaR and check if the subadditivity axiom is satisfied.

- CVaR(X) $= 2$
- CVaR(Y) $= 2.5$
- CVaR (X+Y) $= ?$

Condition Value at Risk: Stochastic formulation

- In order to define the CVaR, we include in our basic model some additional variables and constraints
- In particular, we define the variables L^s

$$L^s = \max(0, K - W^s)$$

or equivalently

$$L^s \geq K - W^s$$

$$L^s \geq 0$$

- To define the objective function we introduce some support variables y^s defined as

$$y^s = \max(0, L^s - VaR)$$

$$y^s \geq L^s - VaR, \quad y^s \geq 0$$

Condition Value at Risk: Stochastic formulation

- Note that VaR is a decision variable here
- The CVaR can be written as

$$CVaR = VaR + \frac{1}{(1-\alpha)} \sum_{s=1}^{S} p_s y^s$$

The overall model

$$\min CVaR = VaR + \frac{1}{(1-\alpha)} \sum_{s=1}^{S} p_s y^s$$

$$\sum_{i=1}^{N} P_{i0} x_i = B$$

$$\sum_{i=1}^{N} \bar{P}_i x_i \geq \theta B$$

$$L^s \geq K - W^s \qquad s = 1, \dots, S$$

$$y^s \geq L^s - VaR \qquad s = 1, \dots, S$$

$$L^s \geq 0 \quad y^s \geq 0 \quad s = 1, \dots, S$$

$$x_i \geq 0 \quad i = 1, \dots, N \quad VAR \geq 0$$