DATA SCIENCE

Multi-Stage Stochastic Programming

Antonio VIOLI

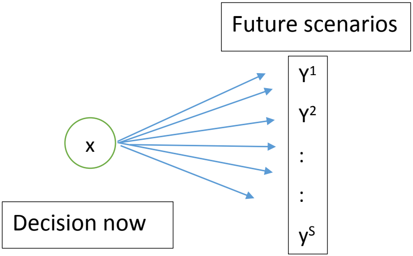
antonio.violi@unirc.it

Introduction

- Multistage stochastic programming formulations arise naturally as a generalization of two-stage models.
- In the two-stage, the decision process can be represented as follows

$$x \xrightarrow{\omega} y(\omega)$$

- In the discrete case



From two-stage to multi-stage

Recall the two-stage formulation introduced in our previous lesson

$$\min \quad z = c^T x + \sum_{s=1}^{S} p_s q^{sT} y^s$$

$$Ax = b$$

$$T^s x + W^s y^s = h^s \qquad s = 1, \dots S$$

$$x \geq 0$$

$$y^s \geq 0 \qquad s = 1, \dots, S$$

Multi-stage stochastic programming

- In the multi-stage case, we consider a planning horizon divided in a given number of stages $t = 1, \ldots, T$
- Thus, the decision process can be represented as

$$x_1 \quad \xrightarrow{\omega_1} x_2 \ldots\ldots x_t \quad \xrightarrow{\omega_t} \quad x_{t+1} \xrightarrow{\omega_{t+2}} \quad x_{t+2} \ldots\ldots$$

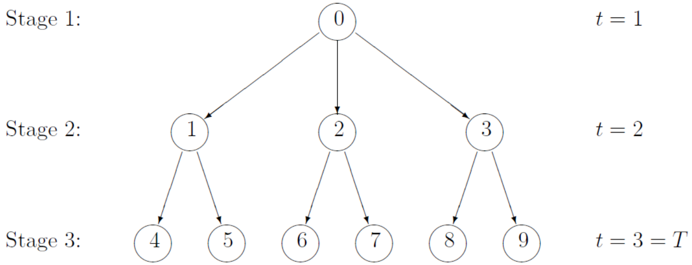
  where
  - $x_t \in R^{n_t}$ denotes the decision taken at stage t
  - ω_t represents the uncertainty whose realizations become known at time t
- REMARKS:
- **stage** is a moment in time, when decisions are taken
- **time period** is a time interval between two stages

Multi-stage stochastic programming

- The multi-stage models can be introduced by considering alternative formulations
- We shall start by introducing the most intuitive one, where the evolution of the uncertain parameters is represented by a scenario tree

The scenario tree

Scenario tree for a three-stage problem



The scenario tree: notation

- The root node 0 is associated with the first stage and refers to deterministic data
- Each node n at level $t \geq 2$ represents a possible realization of the random event ω_t
- Each node n at level t has a unique ancestor (father) at level $t-1$ denoted by $a(n)$ and certain number of successors (children) at level $t+1$
- Nodes with no children are called **leaf nodes**
- In our example, the set of leaf nodes is$\{4, 5, 6, 7, 8\}$
- There is a correspondence between the leaf nodes and the scenarios
- A **scenario** is path from the root node to a leaf node, i.e. it is a joint realization of the random parameters over all the time stages
- Example: Future scenario prices

The scenario tree: notation

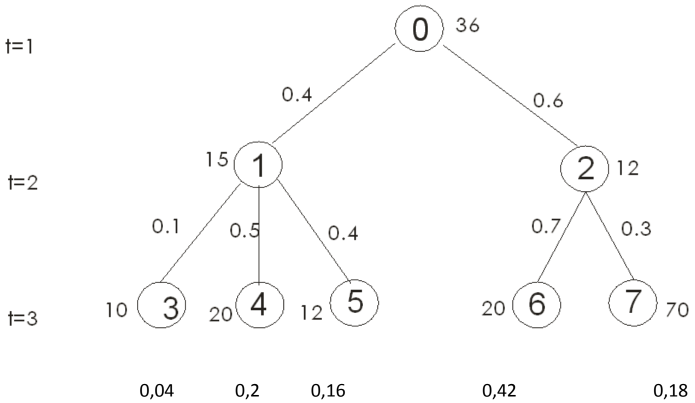
- We denote by $\mathcal{N}$ the set of nodes in the scenario tree
- Let l be the ancestor of node n ($l = a(n)$)
- Let ρ_{ln} be the probability of moving from node l to node n
- Remark:
  - ρ_{ln} is a conditional probability $P(n|l)$
  - $\rho_{ln} \geq 0$ and the sum of the ρ_{ln} associate with the children of l should be 1
- Starting from the ρ_{ln} it is possible to compute the probability associated with each scenario

The scenario tree: notation

- Let n be a generic leaf node
- Let n_1, n_2, n_T the nodes forming the path from the root node to a leaf node
- The probability associated with the scenario is defined by

$$\rho_{n_1 n_2} * \rho_{n_2 n_3} * \cdots * \rho_{n_{T-1} n_T}$$

The scenario tree: example



Remarks

- The branching factor may be arbitrary in principle; the more branches we use, the better our ability to model uncertainty; unfortunately, the number of nodes grows exponentially with the number of stages, as well as the computational effort
- In practice, we are interested in the decisions that must be implemented here and now, i.e., those corresponding to the root node of the tree; the other (recourse) decision variables are instrumental to the aim of devising a robust plan, but they are not implemented in practice, as the multistage model is solved on a rolling horizon basis

Remarks

- This suggests that, in order to model the uncertainty as accurately as possible with a limited computational effort, a possible idea is to branch many paths from the initial node, and less from the subsequent nodes
- There are two basic ways to build a multistage stochastic programming model: the compact model formulation and the split-variable one.

The compact formulation

For each node $n \in \mathcal{N}$ we denote by

- p_n the node probability
- x_n the vector of associated decision variables
- c_n, h^n, T^n, W^n the corresponding matrices and vectors

$$
\begin{aligned}
\min \quad & z = \sum_{n \in \mathcal{N}} p_n c_n^T x_n \\
& Ax_0 = b \\
& T^n x_{a(n)} + W^n x_n = h^n \quad \forall n \in \mathcal{N} - \{0\} \\
& x_n \ \geq 0 \ \ \forall n \in \mathcal{N}
\end{aligned}
$$