DATA SCIENCE

Multistage Formulation of the Dynamic portfolio optimization model

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From Lesson 6 ... The dynamic model

- We consider a planning horizon divided into a number of elementary periods $t = 1, 2, \dots T$
- At each period t of the planning horizon, the investor must decide:
  - The amount of security i to be purchased B_{it}
  - The amount of security i to sell S_{it}
  - The amount of security i to be maintained in the portfolio H_{it}
  - The monetary amount to invest in a risk-free asset v_t

The deterministic model

- **Physical balance constraints** $(i = 1, \dots N)$

$$H_{it} = H_{it-1} + B_{it} - S_{it} \qquad \text{t=2,...,T}$$
$$H_{i1} = InitHold_i + B_{i1} - S_{i1} \quad \text{t=1}$$
$$H_{iT} = 0 \qquad B_{iT} = 0$$
$$(S_{iT} = H_{iT-1})$$

- **Monetary balance constraints**

$$(1-g)\sum_{i=1}^{N} P_{it}S_{it} + F_t + (1+r_t)v_{t-1} =$$
$$(1+g)\sum_{i=1}^{N} P_{it}B_{it} + L_t + v_t \qquad t = 2, \dots, T-1$$
$$(1-g)\sum_{i=1}^{N} P_{i1}S_{i1} + F_1 = (1+g)\sum_{i=1}^{N} P_{i1}B_{i1} + L_1 + v_1$$

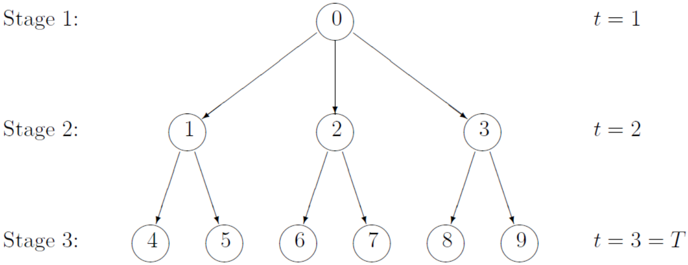
The deterministic model

- **The objective function**

$$max \quad W_T = (1-g)\sum_{i=1}^{N} P_{iT}H_{iT-1} + (1+r_T)v_{T-1} + F_T - L_T$$

The scenario tree

Scenario tree for a three-stage problem



The multi-stage node formulation

- In this case the decision variables are associated with the nodes of the scenario tree which we use to represent the dynamic evolution of the uncertain parameters
- Let $\mathcal{N}$ denote the set of nodes of the scenario tree
- For each node $n \in \mathcal{N}$ we denote by
  - P_{in} the price of asset i
  - L_n the liability
  - F_n the available fund to invest
  - r_n the risk-free interest rate

The multi-stage node formulation

For each node $n \in \mathcal{N}$ we denote by

- S_{in} the amount of security i to sell
- B_{in} the amount of security i to be purchased
- H_{in} the amount of security i to be maintained in the portfolio
- v_n the monetary amount invested in a risk-free asset

The multi-stage node formulation

- **Physical balance constraints ($\forall i$)**

$$H_{in} = H_{ia(n)} + B_{in} - S_{in} \quad \forall n \in \mathcal{N} - \{0\}$$
$$H_{i0} = InitHold_i + B_{i0} - S_{i0}$$
$$H_{in} = 0 \quad B_{in} = 0 \qquad \forall n \in \{\text{leaf node}\}$$
$$S_{in} = H_{ia(n)} \qquad \forall n \in \{\text{leaf node}\}$$

- **Monetary balance constraints**

$$(1-g)\sum_{i=1}^{N} P_{i0}S_{i0} + F_0 = (1+g)\sum_{i=1}^{N} P_{i0}B_{i0} + L_0 + v_0$$
$$(1-g)\sum_{i=1}^{N} P_{in}S_{in} + F_n + (1+r_n)v_{a(n)} =$$
$$(1+g)\sum_{i=1}^{N} P_{in}B_{in} + L_n + v_n \quad n \in \mathcal{N} - \{0\} - \{\text{leaf node}\}$$

The multi-stage node formulation

- **Definition of the wealth at the leaf nodes**

$$W_n = (1 - g)\sum_{i=1}^{N} P_{in} H_{ia(n)} + (1 + r_n) v_{a(n)} + F_n - L_n \; \forall n \text{ leaf node}$$

- **The objective function**

$$\max z = \sum_{n \; \in \{\text{leaf nodes}\}} p_n * W_n$$

The multi-stage split formulation

In this case the decision variables have a double index (stage, scenario)

We denote by

- P_{it}^s the price of asset i at stage t under scenario s
- L_t^s the liability at stage t under scenario s
- F_t^s the available fund to invest at stage t under scenario s
- r_t^s the risk-free interest rate at stage t under scenario s

The multi-stage split formulation

For t and$s$ we denote by

- S_{it}^{s} the amount of security i to sell at stage t under scenario s
- B_{it}^{s} the amount of security i to be purchased at stage t under scenario s
- H_{it}^{s} the amount of security i to be maintained in the portfolio at stage t under scenario s
- v_{t}^{s} the monetary amount invested in a risk-free asset at stage t under scenario s

The multi-stage split formulation

- **Physical balance constraints ($\forall i$)**

$$H^s_{it} = H^s_{it-1} + B^s_{it} - S^s_{it} \qquad t = 2, \dots T-1 \ \forall s$$
$$H^s_{i1} = InitHold_i + B^s_{i1} - S^s_{i1} \qquad s = 1, \dots S$$
$$B^s_{iT} = 0 \quad H^s_{iT} = 0$$
$$S^s_{iT} = H^s_{1T-1} \quad i \ \forall s$$

- **Monetary balance constraints**

$$(1-g)\sum_{i=1}^{N} P^s_{i1} S^s_{i1} + F^s_1 = (1+g)\sum_{i=1}^{N} P^s_{i1} B^s_{i1} + L^s_1 + v^s_1 \ \forall s$$

$$(1-g)\sum_{i=1}^{N} P^s_{it} S^s_{it} + F^s_t + (1+r^s_t)v^s_{t-1} =$$
$$(1+g)\sum_{i=1}^{N} P^s_{it} B^s_{it} + L^s_t + v^s_t \quad t = 2, \dots T-1 \ \forall s$$

The multi-stage node formulation

- **Definition of the final wealth**

$$W_T^s = (1-g)\sum_{i=1}^{N} P_{iT}^s H_{iT-1}^s + (1+r_T^s)v_{T-1}^s + F_T^s - L_T^s \ \forall s$$

- **The non-anticipativity constraints**

$$H_{in} = H_{it(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)$$
$$S_{in} = S_{it(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)$$
$$B_{in} = B_{it(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)$$
$$v_n = v_{t(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)$$

- **The objective function**

$$\max z = \sum_{s=1}^{S} p_s W_T^s$$

The Ito Process for the stock prices

By applying the Ito's Lemma, we get the following solution of the stochastic differential equation

$$P_t = P_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma dW_t}$$

The Wiener process can be rewritten in an approximate form as:

$$\epsilon\sqrt{t} \quad \epsilon \sim N(0,1)$$

Thus

$$P_t = P_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma\epsilon\sqrt{t}}$$

Example

Consider a stock with the following properties:

- volatility $= 30\%$ per annum
- drift (or expected return) $= 15\%$ per year.

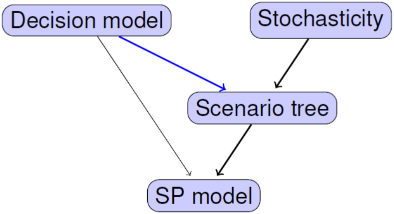
In this case, $\mu = 0.15$ and $\sigma = 0.30$.
Let us assume that the initial price $P_0 = 100$ and that we want to run the simulation on a weekly basis. Thus, Wiener process for the stock price is:

$$P = 100e^{(0.15 - \frac{0.3^2}{2})0.0192 + 0.3\epsilon\sqrt{0.0192}}$$

where $1/52 = 0.0192$ is a conversion factor (year $\rightarrow$ week)

From the theory to the practice

Recall the structure of a stochastic programming model
The choice of a good scenario generation method is problem-dependent



Scenario generation

- Scenario generation represents a critical issue in any stochastic programming formulation since the quality of the generated scenarios affects the efficacy of the solutions provided by the model
- We assume to have available historical data to estimate the distribution of the uncertain parameters
- Intuitively, the higher the number of generated scenarios the more faithful the representation of the uncertainty

Scenario generation

We generally require the scenario set used in a SP to be

- **Comprehensive**
  It should capture all aspects, both extreme and "normal" instances of the underlying distribution
- **Consistent**
  It must capture trends and volatility of the underlying distribution

The first issue is often a question of including "enough" scenarios, but we have to pay attention to the extreme events
The second issue is a matter of making sure that expectation, volatility/correlation of the scenario set is the same of the underlying process

Scenario generation

Different techniques have been proposed in the scientific literature for scenario generation

- Bootstrapping
- MonteCarlo Simulation
- Moment Matching methods

Bootstrapping

The bootstrapping techniques consists in sampling from historical data

- It does not need any distributional assumptions.
- It needs historical data.
- **Main problem: Are historical data a good description of the future?**

MonteCarlo Simulation techniques

The Monte Carlo simulation technique consists in drawing from a given distribution function.
In the case of the stock price, we draw from a normal standard distribution
In our previous example

$$P = 100e^{(0.15 - \frac{0.3^2}{2})0.0192 + 0.3\epsilon\sqrt{0.0192}}$$

If we want to generate two scenarios, we may consider two drawings from the standard normal distribution.
Thus by replacing the values we get

$$P^1 = 100.6882$$

$$P^2 = 96.4811$$

The two-dimensional case

- Let us now consider two different stocks
- We denote by P^1 and P^2 the corresponding prices, whose dynamic behavior is described by

$$dP_t^1 = \mu_1 P_t^1 dt + \sigma_1 P_t^1 dW_t^1$$

$$dP_t^2 = \mu_2 P_t^2 dt + \sigma_2 P_t^2 dW_t^2$$

- We assume that the two processes are correlated and we denote by ρ_{12} the correlation coefficient
- We get

$$P_t^1 = P_0^1 e^{(\mu_1 - \frac{\sigma_1^2}{2})t + \sigma_1 \epsilon_1 \sqrt{t}}$$

$$P_t^2 = P_0^2 e^{(\mu_2 - \frac{\sigma_2^2}{2})t + \sigma_2 \epsilon_2 \sqrt{t}}$$

The two-dimensional case

- The values of ϵ_1 and ϵ_2 are derived starting from two independent drawings x_1 and x_2 from a normal standard distribution. In particular,

$$\epsilon_1 = x_1$$
$$\epsilon_2 = (\rho_{12} x_1 + \sqrt{1 - \rho_{12}^2 x_2})$$

- We may distinguish the following cases
  - independence ($\rho_{12} = 0$)

$$\epsilon_1 = x_1 \quad \epsilon_2 = x_2$$

  - Perfect positive correlation ($\rho_{12} = 1$)

$$\epsilon_1 = x_1 \quad \epsilon_2 = x_1$$

  - Perfect negative correlation ($\rho_{12} = -1$)

$$\epsilon_1 = x_1 \quad \epsilon_2 = -x_1$$

The multidimensional case

In the general case

$$P_t^j = P_o^j e^{(\mu_j - \sigma^2/2)t + \sigma_j \sqrt{t} \sum_{k=1}^{j} C_{jk} x_k}$$

where

$$C_{ij} = \frac{1}{C_{jj}}[\rho_{ij} - \sum_{k=1}^{j-1} C_{jk} C_{ik}] \quad i > j$$

$$C_{jj} = \sqrt{1 - \sum_{k=1}^{j-1} C_{jk}^2}$$

where C are the coefficients of the Cholesky matrix.

The drift and volatility estimation

- Both the drift and the volatility can be determined by the historical returns by applying the well known formula

$$\mu = \frac{\sum_{t=1}^{T} r_t}{T}$$
$$\sigma^2 = \frac{\sum_{t=1}^{T}(r_t - \mu)^2}{T-1}$$

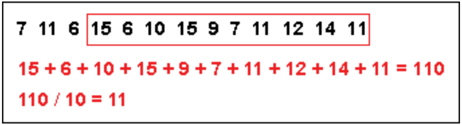
- We observe that the values of μ and σ change over the time and thus "‘very old"’ data could be not useful to generate future scenarios

The simple moving average

To overcome this drawback, it is possible to consider the simple moving average (MA).
Given a set of historical data, we select a sample window of a given size, let say M, and we consider the last M observations

$$\mu_{m1} = \frac{1}{M} \sum_{t=0}^{M-1} r_{T-t}$$

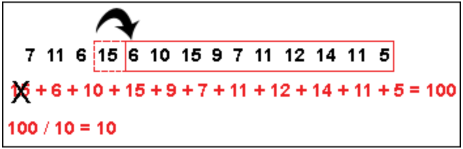
The simple moving average



As new values become available, the oldest data points must be dropped from the set and new data points must come in to replace them.

Thus, the data set is constantly "moving" to account for new data as it becomes available.

This method of calculation ensures that only the current information is being accounted for.



The weighted moving average

- A major drawback of the MA is that it attributes the same weight to all the considered values, whereas it should be more meaningful to attribute higher weights to the more recent observations.
- A weighted average is any average that has multiplying factors to give different weights to data at different positions in the sample window.

$$\mu_{m2} = \frac{\sum_{t=0}^{M-1} \alpha_{T-t} r_{T-t}}{\sum_{t=0}^{M-1} \alpha_{T-t}}$$

The weighted moving average

- In the technical analysis of financial data, a weighted moving average (WMA) has the specific meaning of weights that decrease in arithmetical progression.
  In an M-periods WMA the latest period has weight M, the second latest M - 1, etc., down to one.

$$\mu_{m2} = \frac{\sum_{t=0}^{M-1}(M-t)r_{T-t}}{\sum_{t=1}^{M} t}$$

The weighted moving average

1	2	4	4	5	6	7	8	9	10
15	6	10	15	9	7	11	12	14	11

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